Finite Difference Approximations of SPDEs

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Outline

Stochastic Burgers' Equation

Discretising the Nonlinearity

Simulations

Small noise/viscosity limit

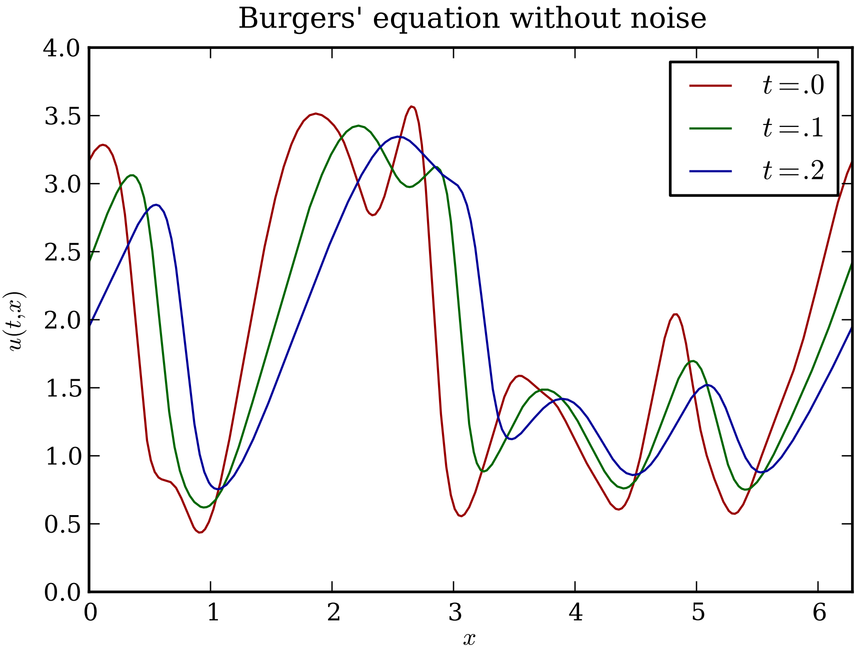
Stochastic Burgers' Equation

We consider the *stochastic Burgers' equation*

$$\partial_t u = \nu \, \partial_x^2 u - u\partial_x u + \sigma \, \partial_t W,$$

where $x \in [0, 2\pi]$, $t \geq 0$, the operator ∂_x^2 is equipped with periodic boundary conditions, and $\partial_t W$ is space time white noise.

- $u\partial_x u$ is a transport term (shifts right if $u > 0$ and left if $u < 0$).
- the $\partial_x^2 u$-term "smoothes" the solution
- $\partial_t W$ adds noise



Burgers' equation without noise
$t=.0$
$t=.1$
$t=.2$
$u(t,x)$
x
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
0
1
2
3
4
5
6

Since the noise W is very "rough", we need to use the concept of *mild solutions*.

Definition. A process u with values $u(t) \in L^2\big([0, 2\pi], \mathbb{R}\big)$ is a solution of the stochastic Burgers' equation, if

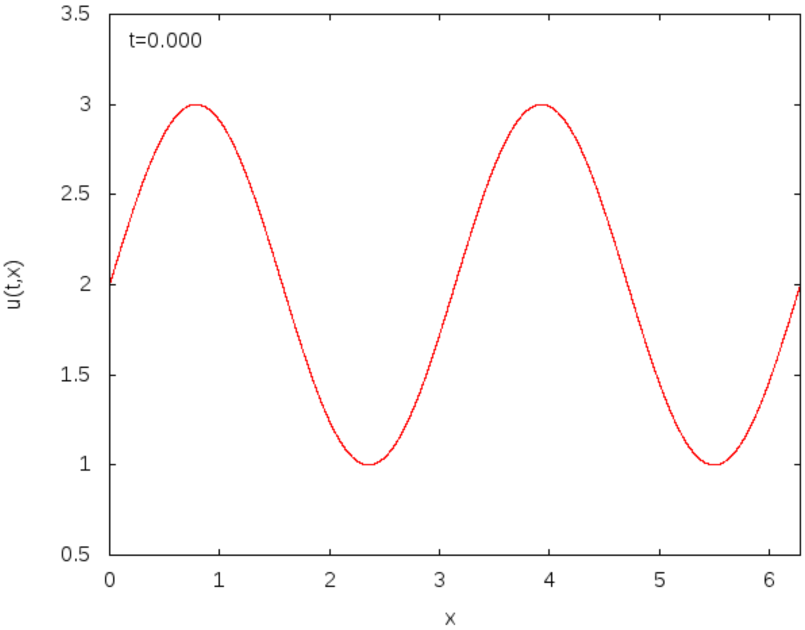
$$u(t) = \mathrm{e}^{-tL} u_0 + \int_0^t \mathrm{e}^{(t-s)L} F\big(u(s)\big)\, ds + \int_0^t \mathrm{e}^{(t-s)L}\, dW_s$$

for all $t \geq 0$, where

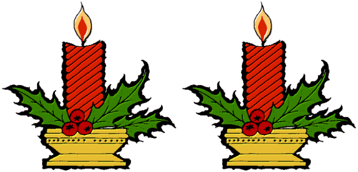
- $L = \partial_x^2$ on $L^2\big([0, 2\pi], \mathbb{R}\big)$ with periodic boundary conditions
- $F(u) = -u\,\partial_x u$
- W is a cylindrical Wiener process

One can prove that there exists a global mild solution in $L^2([0,2\pi],\mathbb{R})$:

- Existence of local solutions follows from abstract theory.
- Existence of global solutions uses the fact that the non-linear drift term can be written as $u\partial_x u = \partial_x(u^2/2)$.
- The solution satisfies $u(t) \in H^{1/2-\varepsilon}$ for all $\varepsilon > 0$.


t=0.000
3.5
3
2.5
2
1.5
1
0.5
u(t,x)
0
1
2
3
4
5
6
x

Discretising the Nonlinearity



To discretise $u\partial_x u$, we consider the approximating equation

$$du_\varepsilon(x,t) = \nu\, \partial_x^2 u_\varepsilon(x,t)\, dt - u_\varepsilon(x,t) D_\varepsilon u_\varepsilon(x,t)\, dt + dW(t),$$

where we define the approximate derivative D_ε by

$$D_\varepsilon u(x,t) = \frac{u(x+a\varepsilon,t) - u(x-b\varepsilon,t)}{(a+b)\varepsilon}$$

for some $a, b \geq 0$ with $a + b > 0$.

- In the absence of the noise term, one can see that this solution converges to the exact solution as $\varepsilon \downarrow 0$.
- For the stochastic equation, we will show that this is not always the case (only for $a = b$).

Consider first the solution v to the stochastic heat equation

$$dv = \nu\, \partial_x^2 v\, dt + dW(t).$$

Using the ansatz

$$v(t,x) = \sum_{n\in\mathbb{Z}} c_n(t) \frac{\mathrm{e}^{inx}}{\sqrt{2\pi}},$$

it can be checked that the stationary solution is given by

$$v(t,x) = \sum_{n\neq 0} \frac{\xi_n(t)}{2\sqrt{\nu\pi}\, in} \mathrm{e}^{inx} + B(t) \frac{1}{\sqrt{2\pi}},$$

where the ξ_n are complex-valued Ornstein-Uhlenbeck processes with $\mathbb{E}|\xi_n(t)|^2 = 1$ and time constant νn^2 that are independent, except for the condition that $\xi_{-n} = \bar{\xi}_n$.

We would expect v to have the same smoothness properties as u.

The derivative of v is then (formally)

$$\partial_x v(x) = \sum_{n \neq 0} \frac{\xi_n(t)}{2\sqrt{\nu\pi}} \mathrm{e}^{inx}.$$

The ε-approximation to the derivative (as defined above) is

$$D_\varepsilon v(x) = \sum_{n \neq 0} \frac{\xi_n(t)}{2\sqrt{\nu\pi}} \frac{\mathrm{e}^{ina\varepsilon} - \mathrm{e}^{-inb\varepsilon}}{in(a+b)\varepsilon} \mathrm{e}^{inx}.$$

It is clear that the terms in approximate derivative are a good approximation only up to $n \approx 1/\varepsilon$.

Comparison of $v\partial_x v$ and $vD_\varepsilon v$

- Since

$$\int_0^{2\pi} \frac{\mathrm{e}^{-i0x}}{\sqrt{2\pi}}\, v(x)\partial_x v(x)\, dx = \frac{1}{\sqrt{2\pi}}\left(\frac{v^2}{2}(2\pi) - \frac{v^2}{2}(0)\right) = 0,$$

the $n = 0$ mode of $v\partial_x v$ vanishes.

- The $n = 0$ mode of $vD_\varepsilon v$ can be found as

$$\sum_{k\neq 0} \frac{\xi_k(t)}{2\sqrt{\nu\pi}ik}\, \frac{\xi_{-k}(t)\left(\mathrm{e}^{-ika\varepsilon} - \mathrm{e}^{ikb\varepsilon}\right)}{2\sqrt{\nu\pi}i(-k)(a+b)\varepsilon} = \sum_{k>0} \frac{|\xi_k(t)|^2}{2\pi\nu k}\, \frac{\cos kb\varepsilon - \cos ka\varepsilon}{(a+b)\varepsilon k}$$

which does not vanish in general ...

… Indeed, as $\varepsilon \to 0$, the expectation of the $n = 0$-mode

$$\sum_{k>0} \frac{|\xi_k(t)|^2}{2\pi\nu k} \frac{\cos kb\varepsilon - \cos ka\varepsilon}{(a+b)\varepsilon k}$$

converges to

$$\frac{1}{2\nu\pi} \int_0^\infty \frac{\cos bx - \cos ax}{(a+b)x^2}\, dx = \frac{1}{4\nu} \frac{b-a}{b+a},$$

which vanishes if and only if $a = b$.

Conjecture

As $\varepsilon \downarrow 0$, the solution of the approximating equation

$$\partial_t u_\varepsilon = \nu\, \partial_x^2 u_\varepsilon - u_\varepsilon D_\varepsilon u_\varepsilon + \sigma\, \partial_t dW(t),$$

converges to the solution of

$$\partial_t u = \nu\, \partial_x^2 u - u\partial_x u - \frac{\sigma^2}{4\nu}\frac{b-a}{b+a} + \sigma\, \partial_t dW(t).$$

Simulations

We will “verify” the conjecture using a numerical simulation. In order to do so, we . . .

- approximate space by

$$\{\Delta x, 2\Delta x, \ldots, (N-1)\Delta x, N\Delta x\}$$

  where $\Delta x = 2\pi/N$ for some $N \in \mathbb{N}$.
- approximate ∂_x^2 by

$$\partial_x^2 u \approx \frac{1}{\Delta x^2} \begin{pmatrix} -2 & 1 & & & 1 \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ 1 & & & 1 & -2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \\ u_n \end{pmatrix} =: Lu$$

- approximate $uD_\varepsilon u$ by

$$\Big(u_j \frac{u_{j+a} - u_{j-b}}{(a+b)\Delta x}\Big)_{j=1,\dots,N}$$

  where $a, b \in \{0, 1\}$.
- approximate dW by $\frac{1}{\sqrt{\Delta x}} dB$ where B is an N-dimensional standard Brownian motion.

The analysis we did above for $\partial_x^2 u - uD_\varepsilon u$ can be repeated for the fully space-discretised equation with drift $Lu - uD_\varepsilon u$.

- The eigenvectors of L are given by e^{inx} where $n \in \{\frac{1}{2}, 1\frac{1}{2}, \ldots, N - \frac{1}{2}\}$ and x is on the grid. The corresponding eigenvalues are

$$\lambda_n = \frac{2\cos(n\Delta x - 1)}{\Delta x^2}.$$

- We can work out the evolution equation of the Fourier modes as before to get the variances in stationarity.
- The same argument as above allows to compute the correction term for the drift. Result: for $a = 1$ and $b = 0$, the additional drift term is $-\sigma^2/4\nu$.

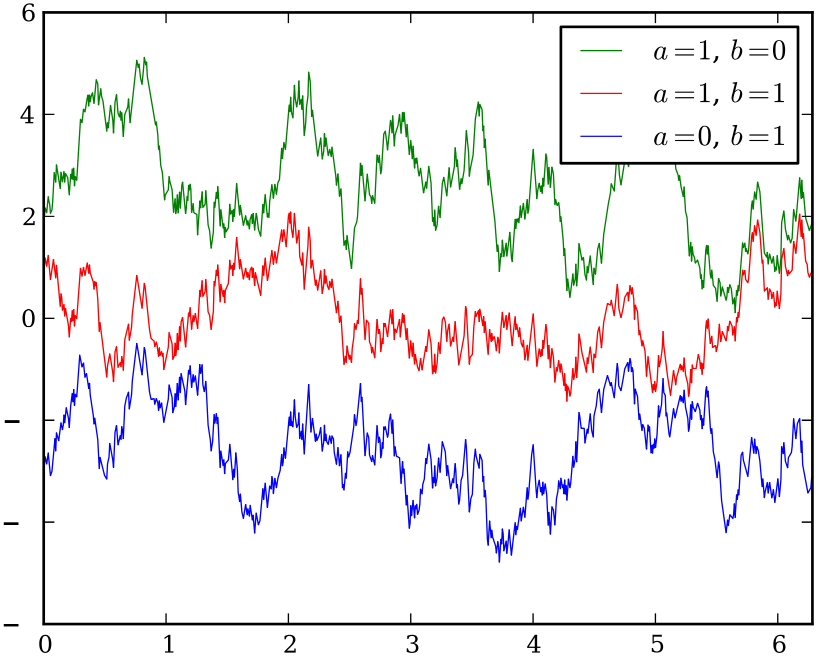
Time Discretisation

We use θ-method with $\theta = 1/2$ to discretise time. Thus, we have

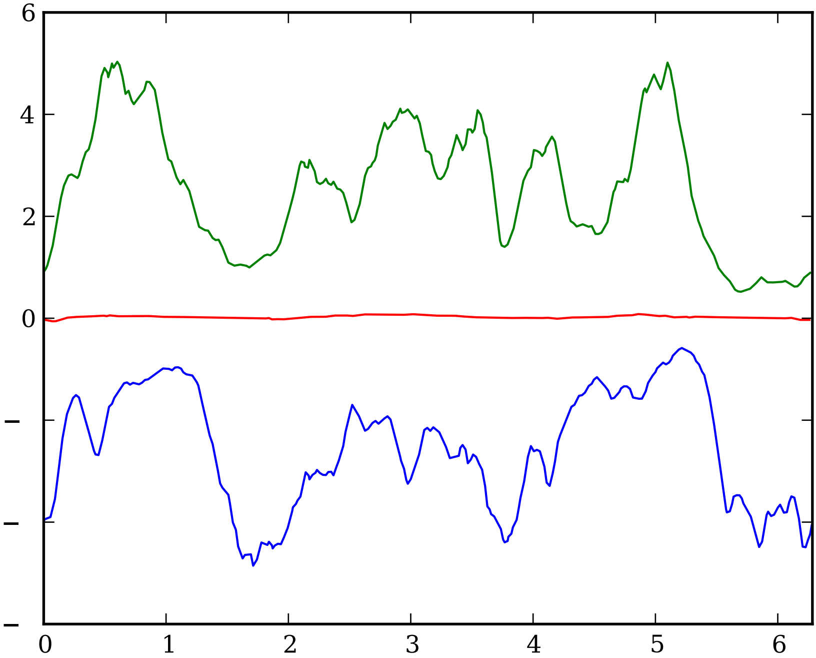
$$x^{(n+1)} = x^{(n)} + \frac{1}{\sigma^2} L\big(\theta x^{(n+1)} + (1-\theta)x^{(n)}\big)\,\Delta t + \text{drift}(x^{(n)})\Delta t + \sqrt{\frac{\Delta t}{\Delta x}}\,\xi^{(n+1)}.$$

Solving this for $x^{(n+1)}$ gives

$$\big(I - \frac{\theta\Delta t}{\sigma^2}L\big)x^{(n+1)} = \big(I + \frac{(1-\theta)\Delta t}{\sigma^2}L\big)x^{(n)} + \text{drift}(x^{(n)})\Delta t + \sqrt{\frac{\Delta t}{\Delta x}}\,\xi^{(n+1)}.$$



$a=1,\ b=0$
$a=1,\ b=1$
$a=0,\ b=1$
6
4
2
0
0
1
2
3
4
5
6


6
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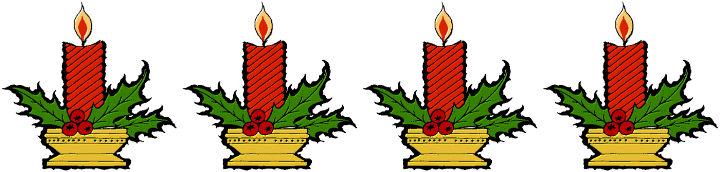
Small noise/viscosity limit

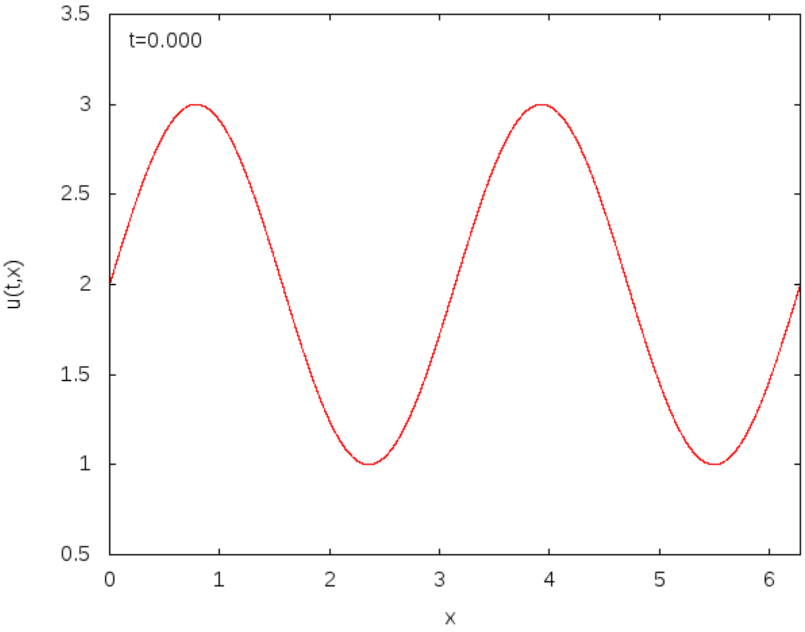


We are interested in the equation

$$du = \varepsilon\, \partial_x^2 u\, dt - u \partial_x u\, dt + \sqrt{\varepsilon}\, dW$$

for $\varepsilon \ll 1$.

- For small ε the solution has “shocks” of width $\mathcal{O}(\varepsilon)$.
- In the limit $\varepsilon = 0$, the centred discretisation for the deterministic equation is unstable.


t=0.000
u(t,x)
x
3.5
3
2.5
2
1.5
1
0.5
0
1
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4
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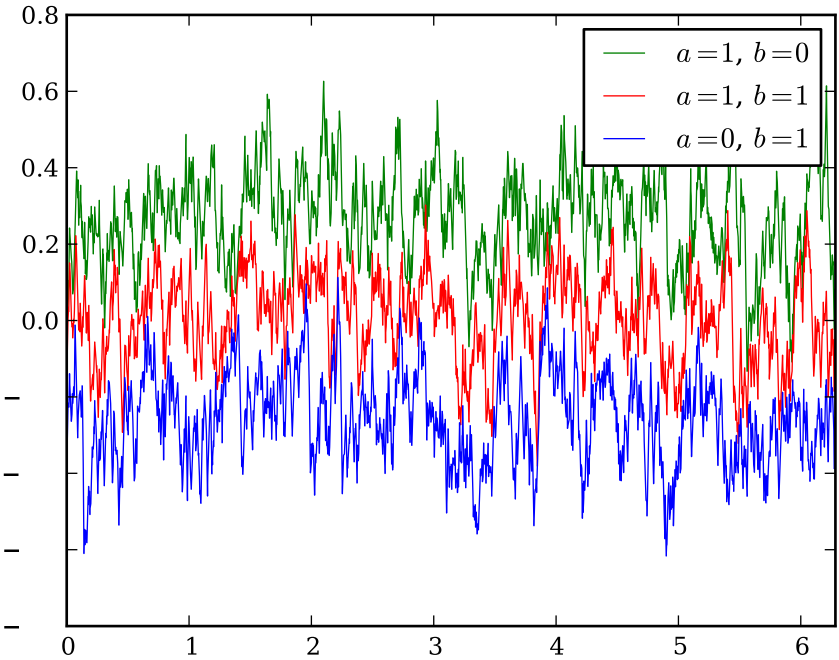
6

Conjecture

$$\partial_t u_n = -u\partial_x u - \frac{c\sigma^2}{4} \,, \qquad (1)$$

1. $\Delta x \ll \varepsilon$: the solution to the finite difference approximation converges to the viscosity solution of (1) where $c \in \{1, 0, -1\}$ depending on whether the discretisation is right-handed, centred, or left-handed.
2. $\varepsilon \ll \Delta x \ll \sqrt{\varepsilon}$: we expect the finite difference approximation to converge to the solutions to (1) only up to the formation of the first shock. After this, one expects to see the solution to become unstable.
3. $\sqrt{\varepsilon} \ll \Delta x$: we expect both the viscosity and the noise term to become irrelevant, so that the solution behaves like the corresponding approximation to the inviscid Burgers' equation.


0.8
0.6
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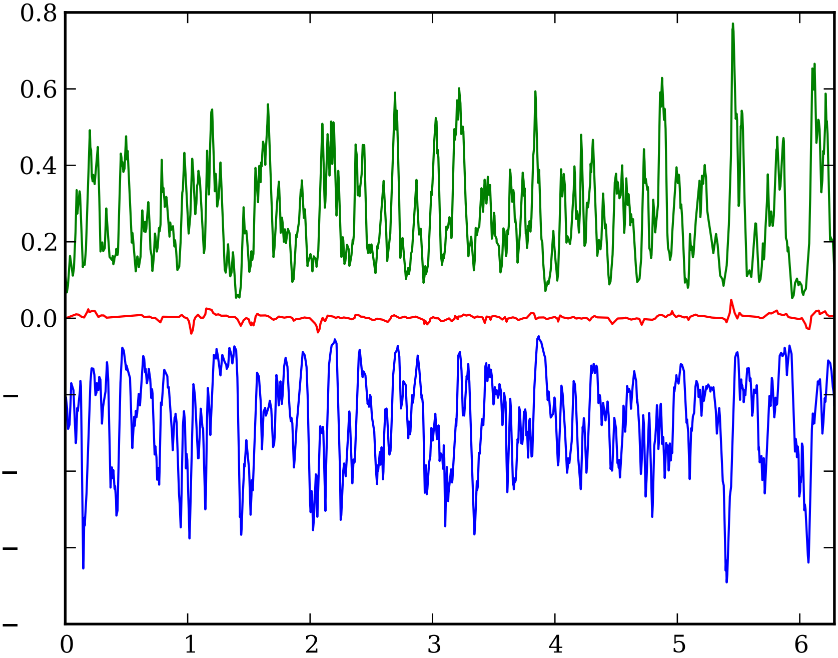
$a=1,\ b=0$
$a=1,\ b=1$
$a=0,\ b=1$



0.8
0.6
0.4
0.2
0.0
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Conclusion

- Finite difference discretisation cans converge to the wrong solution!
- Using the heuristic method presented here, it is sometimes possible to guess the exact form of the error.
- There are various extension possible, *e.g.* more general non-linearities of the form

$$\partial_t u_i = \nu\, \partial_x^2 u_i + \sum_j g_{ij}(u) \partial_x u_j + \sigma\, \partial_t W_i$$

(done) and multiplicative noise (still to do).